

Maximum Principle for Partial Observed Zero-Sum Stochastic Differential Game of Mean-Field SDEs *

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Abstract

In this paper, we consider a partial observed two-person zero-sum stochastic differential game problem where the system is governed by a stochastic differential equation of mean-field type. Under standard assumptions on the coefficients, the maximum principles for optimal open-loop control in a strong sense as well as a weak one are established by the associated optimal control theory in Tang and Meng (2016). To illustrate the general results, a class of linear quadratic stochastic differential game problem is discussed and the existence and dual characterization for the partially observed open-loop saddle are obtained.

1 Introduction

Recently, thanks to many practical applications such as in economics and finance, stochastic optimal control problems for stochastic differential equation (SDE) of mean-field type have been extensively studied and a number of important theoretical and practical application results are obtained under full observations or partial observations. As Djehiche and Tembine (2016) stated in their paper, the speciality of the optimal control problem of mean-field type is that the coefficients of the stochastic state equation and cost functional are dependent not only on the state and the control, but also on their probability distribution. The presence of the mean-field term makes the control problem time-inconsistent and the dynamic programming principle (DPP) ineffective, which motivates to establish the stochastic maximum principle (SMP) to solve this type of optimal control problems instead of trying extensions of DPP. And the so-called full observed optimal control problem is that in this case the controller has full complete information filtration available on the admissible control. We refer to interested readers to Andersson and Djehiche (2011), Buckdahn et al (2011), Li (2012), Meyer-Brandis et al(2012), Shen and Siu (2013), Du et al (2013), Elliott(2013), Hafayed (2013), Yong (2013), Chala(2014), Shen et al (2014), Meng and Shen(2015) and the reference therein for the various optimal control theory results on the mean-field models under full observation. On the other hand, for the partial observed or partial information optimal control problem, the objective is to find an optimal control for which the controller has less information than the complete information filtration. In particular, sometimes an economic model in which there are information gaps among economic agents can be formulated as a partial information optimal control problem (see Øksendal, B. (2006), Kohlmann and Xiong (2007)). A great of results on stochastic optimal control without mean-field term under partial observation or partial information have been obtained by many authors for various types of stochastic systems via establishing the corresponding MP and DPP. See e.g., Bensoussan(1983), Tang (1998), Bagheri et al. (2007), Wu(2010), Wang and Wu(2009), Wang et al(2013, 2015a), and the reference therein for more detailed discussion. Recently, for partial observed stochastic optimal control problem of stochastic systems of mean-field type, due to the theoretical and practical interest, it become more popular, e.g, Wang et al (2014a, 2014b, 2015b, 2016), Djehiche and Tembine (2016), Ma

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and Liu (2017), where the corresponding maximum principles are established and practical finance applications are illustrated.

This purpose of this paper is to extend the optimal control problem to two-person zero-sum differential game problem for SDE of mean-field type under partial observation. As stated in Basar (2010), game theory deals with strategic interactions among multiple decision makers, called players (and in some context agents), with each players preference ordering among multiple alternatives captured in an objective function for that player, which she either tries to maximize (in which case the objective function is a utility function or benefit function) or minimize (in which case we refer to the objective function as a cost function or a loss function). Specially, we say that a game problem has the zero-sum property which means that there is a single performance criterion which one player tries to minimize and the other tries to maximize. Differential game theory investigates conflict problems in systems which are driven by differential equations. This topic lies at the intersection of game theory (several players are involved) and of controlled systems (the differential equations are controlled by the players). The theory of differential games was initiated by Issacs(1954). It was later studied in greater detail by Fleming and Berkovitz (1955). After the development of Pontryagins maximum principle, it became clear that there was a connection between differential games and optimal control theory. In fact, differential game problems represent a generalization of optimal control problems in cases where there are more than one controller or player. However, differential games are conceptually far more complex than optimal control problems in the sense that it is no longer obvious what constitutes a solution; indeed, there are a number of different types of solutions such as minimax solutions for zerosum differential games, Nash solutions for nonzero-sum game problems , Stackelberg differential games, along with possibilities of cooperation and bargaining, see Basar and Olsder (1999), Sethi and Thompson(2000). For the partial information stochastic differential games, recently, An and Øksendal (2008) established a maximum principle for forward systems with Poisson jumps. Moreover, we refer to Wang and Yu (2010, 2012), Meng and Tang(2010), Xiong et al (2016) and the references therein for more associated results on the partial information or partial observed differential games for all kinds of different stochastic systems without mean-field terms.

But to our best knowledge, there is few discussions on the partial observed stochastic differential games problem for the stochastic system of mean-field type, which motives us to write this paper. The main contribution of this paper is to establish the partial observed maximum principle in the weak formulation and strong formulation for the optimal open-loop control under our two-person zero-sum differential games framework by using the results in Tang and Meng (2016). The results obtained in this paper can be considered as a generalization of stochastic optimal control problem of mean-field type to the two-person zero-sum case under partial observations. As an application, a two-person zero-sum stochastic differential game of linear mean-field stochastic differential equations with a quadratic cost criteria under partial observation is discussed where the existence of the corresponding open-loop saddle is obtained and the optimal control is characterized explicitly by adjoint processes.

The rest of this paper is organized as follows. We introduce useful notations and formulate the two-person zero-sum differential game problem for mean-field SDEs under partial observation. Section 2 is devoted to the necessary maximum principle in a weak formulation for optimal open-loop control by the optimal control theory in Tang and Meng (2016). In Section 3, necessary as well as sufficient optimality conditions for our differential games in a strong formulation is derived. As an application, a class of linear quadratic stochastic differential game problem under partial observations is studied.

2 Basic Notations

In this section, we introduce some basic notations which will be used in this paper. Let $\mathcal{T} := [0, T]$ denote a finite time index, where $0 < T < \infty$. We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with two one-dimensional standard Brownian motions $\{W(t), t \in \mathcal{T}\}$ and $\{Y(t), t \in \mathcal{T}\}$, respectively. Let $\{\mathcal{F}_t^W\}_{t \in \mathcal{T}}$ and $\{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}$ be \mathbb{P} -completed natural filtration generated by $\{W(t), t \in \mathcal{T}\}$ and $\{Y(t), t \in \mathcal{T}\}$, respectively. Set $\{\mathcal{F}_t\}_{t \in \mathcal{T}} := \{\mathcal{F}_t^W\}_{t \in \mathcal{T}} \vee \{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}$, $\mathcal{F} = \mathcal{F}_T$. Denote by $\mathbb{E}[\cdot]$ the expectation under the probability \mathbb{P} . Let E be a Euclidean space. The inner product in E is denoted by $\langle \cdot, \cdot \rangle$, and the norm in E is denoted by $|\cdot|$. Let A^\top denote the transpose of the matrix or vector A . For a function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$, denote by ϕ_x its gradient. If $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ (with $k \geq 2$), then $\phi_x = (\frac{\partial \phi_i}{\partial x_j})$ is the corresponding $k \times n$ -Jacobian matrix. By \mathcal{P} we denote the predictable σ field on $\Omega \times [0, T]$ and by $\mathcal{B}(\Lambda)$ the Borel σ -algebra of any topological space Λ . In the follows, K represents a generic

constant, which can be different from line to line.

Next we introduce some spaces of random variable and stochastic processes. For any $\alpha, \beta \in [1, \infty)$, we let

- $M_{\mathcal{F}}^{\beta}(0, T; E)$: the space of all E -valued and \mathcal{F}_t -adapted processes $f = \{f(t, \omega), (t, \omega) \in \mathcal{T} \times \Omega\}$ satisfying

$$\|f\|_{M_{\mathcal{F}}^{\beta}(0, T; E)} \triangleq \left(\mathbb{E} \left[\int_0^T |f(t)|^{\beta} dt \right] \right)^{\frac{1}{\beta}} < \infty.$$

- $S_{\mathcal{F}}^{\beta}(0, T; E)$: the space of all E -valued and \mathcal{F}_t -adapted càdlàg processes $f = \{f(t, \omega), (t, \omega) \in \mathcal{T} \times \Omega\}$

$$\text{satisfying } \|f\|_{S_{\mathcal{F}}^{\beta}(0, T; E)} \triangleq \left(\mathbb{E} \left[\sup_{t \in \mathcal{T}} |f(t)|^{\beta} \right] \right)^{\frac{1}{\beta}} < +\infty.$$

- $L^{\beta}(\Omega, \mathcal{F}, P; E)$: the space of all E -valued random variables ξ on (Ω, \mathcal{F}, P) satisfying $\|\xi\|_{L^{\beta}(\Omega, \mathcal{F}, P; E)} \triangleq \sqrt{\mathbb{E}|\xi|^{\beta}} < \infty$.

- $M_{\mathcal{F}}^{\beta}(0, T; L^{\alpha}(0, T; E))$: the space of all $L^{\alpha}(0, T; E)$ -valued and \mathcal{F}_t -adapted processes $f = \{f(t, \omega), (t, \omega) \in [0, T] \times \Omega\}$ satisfying $\|f\|_{\alpha, \beta} \triangleq \left\{ \mathbb{E} \left[\left(\int_0^T |f(t)|^{\alpha} dt \right)^{\frac{\beta}{\alpha}} \right] \right\}^{\frac{1}{\beta}} < \infty$.

2.1 Formulation of Two-Person Zero-Sum Differential Games of Mean-Field Type Under Partial Observation

In the following, we formulate a partial observed two-person zero-sum stochastic differential game problem in a weak form and a strong form, respectively, where the system is governed by the following nonlinear mean-field stochastic differential equation

$$\begin{cases} dx(t) &= b(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)])dt \\ &\quad + g(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)])dW(t) \\ &\quad + \tilde{g}(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)])dW^{(u_1, u_2)}(t), \\ x(0) &= a \in \mathbb{R}^n, \end{cases} \quad (2.1)$$

with an obervation

$$\begin{cases} dY(t) &= h(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)])dt + dW^{(u_1, u_2)}(t), \\ y(0) &= 0, \end{cases} \quad (2.2)$$

where $b : \mathcal{T} \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U_1 \times U_1 \times U_2 \times U_2 \rightarrow \mathbb{R}^n$, $g : \mathcal{T} \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U_1 \times U_1 \times U_2 \times U_2 \rightarrow \mathbb{R}^n$, $\tilde{g} : \mathcal{T} \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U_1 \times U_1 \times U_2 \times U_2 \rightarrow \mathbb{R}^n$, $h : \mathcal{T} \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U_1 \times U_1 \times U_2 \times U_2 \rightarrow \mathbb{R}$, are given random mapping with $U_1 \subset \mathbb{R}^{k_1}$ and $U_2 \subset \mathbb{R}^{k_2}$ being two given nonempty convex sets. In the above, $u_1(\cdot)$ and $u_2(\cdot)$ are partial observed stochastic processes called admissible control processes of Player 1 and Player 2 respectively defined as follows.

Definition 2.1. For $i = 1, 2$, a partial observed admissible control process for Player i is defined as a stochastic process $u_i : \mathcal{T} \times \Omega \rightarrow U_i$ which is $\{\mathcal{F}_t^Y\}_{t \in \mathcal{T}}$ -adapted and satisfies

$$\mathbb{E} \left[\left(\int_0^T |u_i(t)|^2 dt \right)^2 \right] < \infty. \quad (2.3)$$

The set of all admissible controls is denoted by $\mathcal{A}_i^W, i = 1, 2$. The control pair $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1^W \times \mathcal{A}_2^W$ is called a pair of admissible controls of players.

Now we make the following standard assumptions on the coefficients of the equations (2.1) and (2.2).

Assumption 2.1. The coefficients b, g, \tilde{g} and h are $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U_1) \otimes \mathcal{B}(U_1) \otimes \mathcal{B}(U_2) \otimes \mathcal{B}(U_2)$ -measurable. For each $(x, y, u_1, v_1, u_2, v_2) \in \mathbb{R}^n \times \mathbb{R}^n \times U_1 \times U_1 \times U_2 \times U_2$, $b(\cdot, x, y, u_1, v_1, u_2, v_2), g(\cdot, x, y, u_1, v_1, u_2, v_2),$

$\tilde{g}(\cdot, x, y, u_1, v_1, u_2, v_2)$ and $h(\cdot, x, y, u_1, v_1, u_2, v_2)$ are all $\{\mathcal{F}_t\}_{t \in \mathcal{T}}$ -adapted processes. For almost all $(t, \omega) \in \mathcal{T} \times \Omega$, the mapping

$$(x, y, u_1, v_1, u_2, v_2) \rightarrow \varphi(t, \omega, x, y, u_1, v_1, u_2, v_2)$$

is continuous differentiable with respect to $(x, y, u_1, v_1, u_2, v_2)$ with appropriate growths, where $\varphi = b, g, \tilde{g}$ and h . More precisely, there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^n, u_1, v_1 \in U_1, u_2, v_2 \in U_2$ and a.e. $(t, \omega) \in \mathcal{T} \times \Omega$,

$$\begin{cases} (1 + |x| + |y| + |u_1| + |v_1| + |u_2| + |v_2|)^{-1} |\phi(t, x, y, u_1, v_1, u_2, v_2)| + |\phi_x(t, x, y, u_1, v_1, u_2, v_2)| \\ + |\phi_y(t, x, y, u_1, v_1, u_2, v_2)| + |\phi_{u_1}(t, x, y, u_1, v_1, u_2, v_2)| + |\phi_{v_1}(t, x, y, u_1, v_1, u_2, v_2)| \\ + |\phi_{u_2}(t, x, y, u_1, v_1, u_2, v_2)| + |\phi_{v_2}(t, x, y, u_1, v_1, u_2, v_2)| \leq C, \varphi = b, g, \tilde{g}; \\ |h(t, x, y, u_1, v_1, u_2, v_2)| + |h_x(t, x, y, u_1, v_1, u_2, v_2)| + |h_y(t, x, y, u_1, v_1, u_2, v_2)| + |h_{u_1}(t, x, y, u_1, v_1, u_2, v_2)| \\ + |h_{v_1}(t, x, y, u_1, v_1, u_2, v_2)| + |h_{u_2}(t, x, y, u_1, v_1, u_2, v_2)| + |h_{v_2}(t, x, y, u_1, v_1, u_2, v_2)| \leq C. \end{cases}$$

Now under Assumption 2.1, we begin to discuss the well-posedness of (2.1) and (2.2). Indeed, putting (2.2) into the state equation (2.1), we get that

$$\begin{cases} dx(t) &= [(b - \tilde{g}h)(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)])]dt \\ &\quad + g(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)])dW(t) \\ &\quad + \tilde{g}(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)])dY(t), \\ x(0) &= a. \end{cases} \quad (2.4)$$

Under Assumption 2.1, for any $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1^W \times \mathcal{A}_2^W$, by Lemma 1.4 in Tang and Meng (2016), (2.4) admits a strong solution $x(\cdot) \equiv x^{(u_1, u_2)}(\cdot) \in S_{\mathcal{F}}^4(0, T; \mathbb{R}^n)$. On the other hand, for any $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1^W \times \mathcal{A}_2^W$, associated with the corresponding solution $x^{(u_1, u_2)}(\cdot)$ of (2.4), introduce a stochastic process $Z^{(u_1, u_2)}(\cdot)$ defined by the unique solution of the following mean-field SDE

$$\begin{cases} dZ^{(u_1, u_2)}(t) &= Z^{(u_1, u_2)}(t)h(t, x^{(u_1, u_2)}(t), \mathbb{E}[x^{(u_1, u_2)}(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)])dY(t), \\ Z^{(u_1, u_2)}(0) &= 1. \end{cases} \quad (2.5)$$

Define a new probability measure $\mathbb{P}^{(u_1, u_2)}$ on (Ω, \mathcal{F}) by $d\mathbb{P}^{(u_1, u_2)} = Z^{(u_1, u_2)}(1)d\mathbb{P}$. Then from Girsanov's theorem and (2.2), $(W(\cdot), W^{(u_1, u_2)}(\cdot))$ is an \mathbb{R}^2 -valued standard Brownian motion defined in the new probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}^{(u_1, u_2)})$. So $(\mathbb{P}^{(u_1, u_2)}, X^{(u_1, u_2)}(\cdot), Y(\cdot), W(\cdot), W^{(u_1, u_2)}(\cdot))$ is a weak solution on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathcal{T}})$ of (2.1) and (2.2).

Now for any given pair of admissible control $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1^W \times \mathcal{A}_2^W$, and the corresponding weak solution $(\mathbb{P}^{(u_1, u_2)}, x^{(u_1, u_2)}(\cdot), Y(\cdot), W(\cdot), W^{(u_1, u_2)}(\cdot))$ of (2.1) and (2.2), we introduce the following cost functional in the weak form,

$$\begin{aligned} J(u_1(\cdot), u_2(\cdot)) &= \mathbb{E}^{(u_1, u_2)} \left[\int_0^T l(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)])dt \right. \\ &\quad \left. + m(x(T), \mathbb{E}[x(T)]) \right]. \end{aligned} \quad (2.6)$$

where $\mathbb{E}^{(u_1, u_2)}$ denotes the expectation with respect to the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}^{(u_1, u_2)})$ and $l : \mathcal{T} \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U_1 \times U_1 \times U_2 \times U_2 \rightarrow \mathbb{R}$, $m : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ are given random mappings satisfying the following assumption:

Assumption 2.2. l is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(U_1) \otimes \mathcal{B}(U_1) \otimes \mathcal{B}(U_2) \otimes \mathcal{B}(U_2)$ -measurable, and m is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}^n)$ -measurable. For each $(x, y, u_1, v_1, u_2, v_2) \in \mathbb{R}^n \times \mathbb{R}^n \times U_1 \times U_1 \times U_2 \times U_2$, $f(\cdot, x, y, u_1, v_1, u_2, v_2)$ is an \mathbb{F} -adapted process, and $m(x, y)$ is an \mathcal{F}_T -measurable random variable. For almost all $(t, \omega) \in [0, T] \times \Omega$, the mappings

$$(x, y, u_1, v_1, u_2, v_2) \rightarrow l(t, \omega, x, y, u_1, v_1, u_2, v_2)$$

and

$$(x, y) \rightarrow m(\omega, x, y)$$

are continuous differentiable with respect to $(x, y, u_1, v_1, u_2, v_2)$ with appropriate growths, respectively. More precisely, there exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^n$, $u_1, v_1 \in U_1$, $u_2, v_2 \in U_2$ and a.e. $(t, \omega) \in [0, T] \times \Omega$,

$$\begin{cases} (1 + |x| + |y| + |u_1| + |v_1| + |u_2| + |v_2|)^{-1} (|l_x(t, x, y, u_1, v_1, u_2, v_2)| + |l_y(t, x, y, u_1, v_1, u_2, v_2)| \\ \quad + |l_{u_1}(t, x, y, u_1, v_1, u_2, v_2)| + |l_{v_1}(t, x, y, u_1, v_1, u_2, v_2)| + |l_{u_2}(t, x, y, u_1, v_1, u_2, v_2)| + |l_{v_2}(t, x, y, u_1, v_1, u_2, v_2)|) \\ \quad + (1 + |x|^2 + |y|^2 + |u_1|^2 + |v_1|^2 + |u_2|^2 + |v_2|^2)^{-1} |l(t, x, y, u_1, v_1, u_2, v_2)| \leq C; \\ (1 + |x|^2 + |y|^2)^{-1} |m(x, y)| + (1 + |x| + |y|)^{-1} (|m_x(x, y)| + |m_y(x, y)|) \leq C. \end{cases}$$

Under Assumption 2.1 and 2.2, by the estimates in Lemma 1.4 in Tang and Meng (2016), we get that the cost functional is well-defined.

Then we can put forward the following partially observed two-person zero-sum differential game problem in its weak formulation, i.e., with changing the reference probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P}^u)$, as follows.

Problem 2.1. Find a pair of admissible control pair $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ over $\mathcal{A}_1^W \times \mathcal{A}_2^W$ such that

$$J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \sup_{u_2(\cdot) \in \mathcal{A}_2^W} \left(\inf_{u_1(\cdot) \in \mathcal{A}_1^W} J(u_1(\cdot), u_2(\cdot)) \right) = \inf_{u_1^W(\cdot) \in \mathcal{A}_1^W} \left(\inf_{u_2(\cdot) \in \mathcal{A}_2^W} J(u_1(\cdot), u_2(\cdot)) \right), \quad (2.7)$$

subject to the state equation (2.1), the observation equation (2.2) and the cost functional (2.6).

Obviously, according to Bayes' formula, the cost functional (2.6) can be rewritten as

$$\begin{aligned} J(u_1(\cdot), u_2(\cdot)) = & \mathbb{E} \left[\int_0^T Z^{(u_1, u_2)}(t) l(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)]) dt \right. \\ & \left. + Z^{(u_1, u_2)}(T) m(x(T), \mathbb{E}[x(T)]) \right]. \end{aligned} \quad (2.8)$$

Therefore, we can translate Problem 2.1 into the following equivalent optimal control problem in its strong formulation, i.e., without changing the reference probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$, where $Z^{(u_1, u_2)}(\cdot)$ will be regarded as an additional state process besides the state process $x^{(u_1, u_2)}(\cdot)$.

Problem 2.2. Find an admissible open-loop control $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ over $\mathcal{A}_1^W \times \mathcal{A}_2^W$ such that

$$J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \sup_{u_2(\cdot) \in \mathcal{A}_2^W} \left(\inf_{u_1(\cdot) \in \mathcal{A}_1^W} J(u_1(\cdot), u_2(\cdot)) \right) = \inf_{u_1(\cdot) \in \mathcal{A}_1^W} \left(\inf_{u_2(\cdot) \in \mathcal{A}_2^W} J(u_1(\cdot), u_2(\cdot)) \right), \quad (2.9)$$

subject to the cost functional (2.8) and the following state equation

$$\begin{cases} dx(t) &= [(b - \tilde{g}h)(t, x(t), u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)]) dt \\ &\quad + g(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)]) dW(t) \\ &\quad + \tilde{g}(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)]) dY(t), \\ dZ(t) &= Z(t)h(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)]) dY(t), \\ Z(0) &= 1, \\ x(0) &= a \in \mathbb{R}^n. \end{cases} \quad (2.10)$$

Any $(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) \in \mathcal{A}_1^W \times \mathcal{A}_2^W$ satisfying (2.12) is called an optimal open-loop control. The corresponding strong solution $(\bar{x}(\cdot), \bar{Z}(\cdot))$ of (2.10) is called the optimal state process. Then $(\bar{u}_1(\cdot), \bar{u}_2(\cdot); \bar{x}(\cdot), \bar{Z}(\cdot))$ is called an optimal pair.

Roughly speaking, for the zero-sum differential game, Player I seek control $\bar{u}_1(\cdot)$ to minimize (2.8), and Player II seek control $\bar{u}_2(\cdot)$ to maximize (2.8). Therefore, (2.8) represents the cost for Player I and the payoff for Player II. If $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ is an optimal open-loop control, it is easy to check that

$$J(\bar{u}_1(\cdot), u_2(\cdot)) \leq J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) \leq J(u_1(\cdot), \bar{u}_2(\cdot)) \quad (2.11)$$

for all admissible open-loop controls $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$. We refer to $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ as an open-loop saddle. On the other hand, if $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ is an open-loop saddle, by Remark 1.2 of Chapter VI in [6], we know that

$$\sup_{u_2(\cdot) \in \mathcal{A}_2^W} \left(\inf_{u_1(\cdot) \in \mathcal{A}_1^W} J(u_1(\cdot), u_2(\cdot)) \right) = \inf_{u_1(\cdot) \in \mathcal{A}_1^W} \left(\inf_{u_2(\cdot) \in \mathcal{A}_2^W} J(u_1(\cdot), u_2(\cdot)) \right). \quad (2.12)$$

In this paper, provided the original state equation (2.1) and the observation equation (2.2), we will also study the partially observed optimal control problem in its strong formulation, i.e. without changing the reference probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$. Precisely, different from the cost functional (2.6), the cost functional in this case is defined by

$$J(u_1(\cdot), u_2(\cdot)) = \mathbb{E} \left[\int_0^T l(t, X(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)]) dt + m(X(T), \mathbb{E}[x(T)]) \right]. \quad (2.13)$$

Note that $\mathbb{E}(\cdot)$ is the expectation with the original probability \mathbb{P} independent of the control $(u_1(\cdot), u_2(\cdot))$. In this case, different from the partially observed optimal control problem in weak sense discussed before, we do not need require the pair of admissible control processes satisfies (2.3). In this case, a pair of admissible control processes $(u_1(\cdot), u_2(\cdot))$ is defined as a $\{\mathcal{F}_t^Y\}_{0 \leq t \leq T}$ adapted stochastic process valued in $U_1 \times U_2$ satisfying

$$\mathbb{E} \left[\int_0^T |u_1(t)|^2 dt \right] + \mathbb{E} \left[\int_0^T |u_2(t)|^2 dt \right] < \infty. \quad (2.14)$$

The set of all admissible controls in this case is denoted by $\mathcal{A}_1^S \times \mathcal{A}_2^S$.

Then we can put forward the partially observed two-person zero-sum differential game problem in its strong formulation as follows.

Problem 2.3. Find an admissible open-loop control $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ over $\mathcal{A}_1^W \times \mathcal{A}_2^W$ such that

$$J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \sup_{u_2(\cdot) \in \mathcal{A}_2^S} \left(\inf_{u_1(\cdot) \in \mathcal{A}_1^S} J(u_1(\cdot), u_2(\cdot)) \right) = \inf_{u_1(\cdot) \in \mathcal{A}_1^S} \left(\inf_{u_2(\cdot) \in \mathcal{A}_2^S} J(u_1(\cdot), u_2(\cdot)) \right), \quad (2.15)$$

subject to the cost functional (2.8) and the following state equation

$$\begin{cases} dx(t) &= [(b - \tilde{g}h)(t, x(t), u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)]) dt \\ &\quad + g(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)]) dW(t) \\ &\quad + \tilde{g}(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)]) dY(t), \\ x(0) &= a \in \mathbb{R}^n. \end{cases} \quad (2.16)$$

Note that under Assumptions 2.1 and 2.2, for any admissible control $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1^S \times \mathcal{A}_2^S$, by Lemma 1.4 in Tang and Meng (2016), the state (2.10) has a unique solution $x(\cdot) \in S_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$ and $J(u_1(\cdot), u_2(\cdot)) < \infty$, so Problem 2.3 is well-defined.

3 Stochastic Maximum Principle For Zero-Sum Differential Games in Weak Formulation

This section is devoted to establishing the stochastic maximum principle of Problem 2.1 or Problem 2.2, i.e., establishing the necessary optimality condition of Pontryagin's type for a pair of admissible controls to be optimal.

To this end, for the state equation (2.10), we first introduce the corresponding adjoint equation. Actually, define the Hamiltonian function $H : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U_1 \times U_2 \times U_2 \times U_2 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\begin{aligned} H(t, x, y, u_1, v_1, u_2, v_2, p, q, \tilde{q}, \bar{R}) \\ = \langle p, b(t, x, y, u_1, v_1, u_2, v_2) \rangle + \langle q, g(t, x, y, u_1, v_1, u_2, v_2) \rangle + \langle \tilde{q}, \tilde{g}(t, x, y, u_1, v_1, u_2, v_2) \rangle \\ + \bar{R}h(t, x, y, u_1, v_1, u_2, v_2) + l(t, x, y, u_1, v_1, u_2, v_2). \end{aligned} \quad (3.1)$$

For the state equation (2.10) associated with any given admissible pair $(\bar{u}_1(\cdot), \bar{u}_2(\cdot), \bar{x}(\cdot), \bar{Z}(\cdot))$, the corresponding adjoint equation is defined as follows:

$$\left\{ \begin{aligned} d\bar{r}(t) &= -l(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}_1(t), \mathbb{E}[\bar{u}_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)])dt + \bar{R}(t) dW(t) + \bar{\tilde{R}}(t) dW^{\bar{u}}(t), \\ d\bar{p}(t) &= - \left\{ H_x(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}_1(t), \mathbb{E}[\bar{u}_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)]) \right. \\ &\quad \left. + \frac{1}{z^{(u_1, u_2)}(t)} \mathbb{E}^{(u_1, u_2)} [H_y(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}_1(t), \mathbb{E}[\bar{u}_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)])] \right\} dt \\ &\quad + \bar{q}(t) dW(t) + \bar{\tilde{q}}(t) dW^{(\bar{u}_1, \bar{u}_2)}(t), \\ \bar{r}(T) &= m(\bar{x}(T), \mathbb{E}[\bar{x}(T)]), \\ \bar{p}(T) &= m_x(\bar{x}(T), \mathbb{E}[\bar{x}(T)]) + \frac{1}{z^{(u_1, u_2)}(T)} \mathbb{E}^u [m_x(\bar{x}(T), \mathbb{E}[\bar{x}(T)])], \end{aligned} \right. \quad (3.2)$$

where

$$\begin{aligned} H(t, x, y, u_1, v_1, u_2, v_2) \\ =: H(t, x, y, u_1, v_1, u_2, v_2, \bar{p}(t), \bar{q}(t), \bar{\tilde{q}}(t), \bar{R}(t) - \bar{g}(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}_1(t), \mathbb{E}[\bar{u}_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)])^\top \bar{p}(t)). \end{aligned} \quad (3.3)$$

Note the adjoint equation (3.2) is a mean-field backward stochastic differential equation whose solution consists of an 6-tuple process $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot), \bar{r}(\cdot), \bar{R}(\cdot), \bar{\tilde{R}}(\cdot))$. Under Assumptions 2.1 and 2.2, by Buckdahn (2009b), it is easily to see that the adjoint equation (3.2) admits a unique solution $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot), \bar{r}(\cdot), \bar{R}(\cdot), \bar{\tilde{R}}(\cdot)) \in S_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times S_{\mathcal{F}}^2(0, T; \mathbb{R}) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}) \times M_{\mathcal{F}}^2(0, T; \mathbb{R})$, also called the adjoint process corresponding the admissible pair $(\bar{u}(\cdot); \bar{x}(\cdot), \bar{Z}(\cdot))$.

Now we are in a position to state our main result: stochastic maximum principle of Problem 2.1 or 2.2.

Theorem 3.1. *Let assumptions 2.1 and 2.2 be satisfied. Let $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ be an optimal open-loop control of Problem 2.1 or 2.2. Suppose that $(\bar{x}(\cdot), \bar{Z}(\cdot))$ is the state process of the system (2.10) corresponding to $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$. Let $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot), \bar{r}(\cdot), \bar{R}(\cdot), \bar{\tilde{R}}(\cdot))$ be the unique solution of the adjoint equation (3.2) corresponding $(\bar{u}_1(\cdot), \bar{u}_2(\cdot); \bar{x}(\cdot), \bar{Z}(\cdot))$. Then the optimality conditions*

$$\left\langle \mathbb{E}[Z^{(\bar{u}_1, \bar{u}_2)}(t) \bar{H}_{u_1}(t) | \mathcal{F}_t^Y] + \mathbb{E}^{(\bar{u}_1, \bar{u}_2)}[\bar{H}_{v_1}(t)], u_1 - \bar{u}_1(t) \right\rangle \geq 0 \quad (3.4)$$

and

$$\left\langle \mathbb{E}[Z^{(\bar{u}_1, \bar{u}_2)}(t) \bar{H}_{u_2}(t) | \mathcal{F}_t^Y] + \mathbb{E}^{(\bar{u}_1, \bar{u}_2)}[\bar{H}_{v_2}(t)], u_2 - \bar{u}_2(t) \right\rangle \leq 0 \quad (3.5)$$

hold for any $(u_1, u_2) \in U_1 \times U_2$ and a.e. $(t, \omega) \in [0, T] \times \Omega$. Here using the notation (3.3), for $i = 1, 2$, we set

$$\bar{H}_{u_i}(t) = H_u(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}_1(t), \mathbb{E}[\bar{u}_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)]) \quad (3.6)$$

and

$$\bar{H}_{v_i}(t) = H_{v_i}(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)]). \quad (3.7)$$

Proof: Since $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ is an optimal open-loop control, then $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ is an open-loop saddle point, i.e.,

$$J(\bar{u}_1(\cdot), u_2(\cdot)) \leq J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) \leq J(u_1(\cdot), \bar{u}_2(\cdot)). \quad (3.8)$$

So we have

$$J_1(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \min_{u_1(\cdot) \in \mathcal{A}_1^W} J_1(u_1(\cdot), \bar{u}_2(\cdot)) \quad (3.9)$$

and

$$J_2(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \max_{u_2(\cdot) \in \mathcal{A}_2^W} J_2(\bar{u}_1(\cdot), u_2(\cdot)). \quad (3.10)$$

By (3.9), $\bar{u}_1(\cdot)$ can be regarded as an optimal control of an optimal control problem where the controlled system is

$$\begin{cases} dx(t) &= [(b - \tilde{g}h)(t, x(t), u_1(t), \mathbb{E}[u_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)])dt \\ &\quad + g(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)])dW(t) \\ &\quad + \tilde{g}(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)])dY(t), \\ dZ(t) &= Z(t)h(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)])dY(t), \\ Z(0) &= 1, \\ x(0) &= a \in \mathbb{R}^n, \end{cases} \quad (3.11)$$

and the cost functional is

$$\begin{aligned} J(u_1(\cdot), \bar{u}_2(\cdot)) = & \mathbb{E} \left[\int_0^T Z^{(u_1, \bar{u}_2)}(t) l(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)]) dt \right. \\ & \left. + Z^{(u_1, \bar{u}_2)}(T) m(x(T), \mathbb{E}[x(T)]) \right]. \end{aligned} \quad (3.12)$$

Then for this case, it is easy to check that the Hamilton is $H(t, x, y, u_1, v_1, \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)], p, q, \tilde{q}, \tilde{R})$ and for the optimal control $\bar{u}_1(\cdot) \in \mathcal{A}_1^W$, the corresponding optimal state process and the adjoint process is still $\bar{x}(\cdot)$ and $(\bar{p}(\cdot), \bar{q}(\cdot), \tilde{\bar{q}}(\cdot), \bar{r}(\cdot), \bar{R}(\cdot), \tilde{\bar{R}}(\cdot))$, respectively. Thus applying the partial necessary stochastic maximum principle for optimal control problems (see Theorem 2.1 in Tang and Meng (2016)), we can obtain (3.4). Similarly, from (3.10), we can obtain (3.5). The proof is complete.

4 Stochastic Maximum Principle For Zero-Sum Differential Games in Strong Formulation

This section is devoted to establishing the stochastic maximum principles of Problem 2.3. In this case, the Hamiltonian $H : [0, T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times U_1 \times U_1 \times U_2 \times U_2 \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is defined by

$$\begin{aligned} H(t, x, y, u_1, v_1, u_2, v_2, p, q, \tilde{q}) = & \langle p, b(t, x, y, u_1, v_1, u_2, v_2) - \tilde{g}(t, x, y, u_1, v_1, u_2, v_2)h(t, x, y, u_1, v_1, u_2, v_2) \rangle \\ & + \langle q, g(t, x, y, u_1, v_1, u_2, v_2) \rangle + \langle \tilde{q}, \tilde{g}(t, x, y, u_1, v_1, u_2, v_2) \rangle \\ & + l(t, x, y, u_1, v_1, u_2, v_2). \end{aligned} \quad (4.1)$$

Then for any admissible pair $(\bar{u}_1(\cdot), \bar{u}_2(\cdot); \bar{x}(\cdot))$, the corresponding adjoint process is defined as the solution to the following mean-field BSDE:

$$\begin{cases} d\bar{p}(t) = - \left[\bar{H}_x(t) + \mathbb{E}[\bar{H}_y(t)] \right] dt + \bar{q}(t) dW(t) + \tilde{\bar{q}}(t) dY(t) \\ \bar{P}(T) = \bar{m}_x(T) + \mathbb{E}[\bar{m}_y(T)], \end{cases} \quad (4.2)$$

where we have used the following shorthand notation

$$\begin{cases} \bar{H}(t) = H(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}_1(t), \mathbb{E}[\bar{u}_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)], \bar{p}(t), \bar{q}(t), \tilde{\bar{q}}(t)), \\ \bar{m}(T) = m(\bar{x}(T), \mathbb{E}[\bar{x}(T)]). \end{cases} \quad (4.3)$$

Under Assumption 2.1 and 2.2, by Buckdahn (2009), (4.2) admits a unique strong slution $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot)) \in S_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$, which is also called the adjoint process corresponding to the admissible pair $(u_1(\cdot), u_2(\cdot), x(\cdot))$

4.1 Sufficient Conditions of Optimality

In this subsection, we are going to establish the sufficient Pontryagin maximum principle of Problem 2.3.

Theorem 4.1. [Sufficient Stochastic Maximum Principle] *Let Assumptions 2.1 and 2.2 be satisfied. Let $(\bar{u}_1(\cdot), \bar{u}_2(\cdot); \bar{x}(\cdot))$ be an admissible pair and $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot))$ be the unique strong solution of the corresponding adjoint equation (4.2).*

(i) *Suppose that, for all $t \in [0, T]$, $m(x, y)$ is convex in (x, y) , and the mapping*

$$(x, y, u_1, v_1) \mapsto H(t, x, y, u_1, v_1, \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)], \bar{p}(t), \bar{q}(t), \bar{\tilde{q}}(t))$$

is convex. For any $u_1(\cdot) \in \mathcal{A}_1^S$,

$$\mathbb{E} \left[\langle u_1(t) - \bar{u}_1(t), \bar{H}_{u_1}(t) + \mathbb{E}[\bar{H}_{v_1}(t)] \rangle \right] \geq 0. \quad (4.4)$$

Then

$$J(u_1(\cdot), \bar{u}_2(\cdot)) \geq J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)), \quad \text{for all } u_1(\cdot) \in \mathcal{A}_1^S$$

and

$$J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \inf_{u_1(\cdot) \in \mathcal{A}_1^S} J(u_1(\cdot), \bar{u}_2(\cdot)).$$

(ii) *Suppose that, for all $t \in [0, T]$, $m(x, y)$ is concave in (x, y) and*

$$(x, y, u_2, v_2) \mapsto H(t, x, y, \bar{u}_1(t), \mathbb{E}[\bar{u}_1(t)], u_2, v_2, \bar{p}(t), \bar{q}(t), \bar{\tilde{q}}(t))$$

is concave. For any $u_2(\cdot) \in \mathcal{A}_2^S$,

$$\mathbb{E} \left[\langle u_2(t) - \bar{u}_2(t), \bar{H}_{u_2}(t) + \mathbb{E}[\bar{H}_{v_2}(t)] \rangle \right] \leq 0. \quad (4.5)$$

Then

$$J(\bar{u}_1(\cdot), u_2(\cdot)) \leq J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)), \quad \text{for all } u_2(\cdot) \in \mathcal{A}_2^S$$

and

$$J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \sup_{u_2(\cdot) \in \mathcal{A}_2^S} J(\bar{u}_1(\cdot), u_2(\cdot)).$$

(iii) *If both case (i) and (ii) hold (which implies, in particular, that $m(x, y)$ is an affine function), then $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ is an open-loop saddle point and*

$$J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \sup_{u_2(\cdot) \in \mathcal{A}_2^S} \left(\inf_{u_1(\cdot) \in \mathcal{A}_1^S} J(u_1(\cdot), u_2(\cdot)) \right) = \inf_{u_1(\cdot) \in \mathcal{A}_1^S} \left(\sup_{u_2(\cdot) \in \mathcal{A}_2^S} J(u_1(\cdot), u_2(\cdot)) \right). \quad (4.6)$$

Proof. (i) In the following, we consider a stochastic optimal control problem over \mathcal{A}_1^S where the system is

$$\begin{cases} dx(t) &= [(b - \tilde{g}h)(t, x(t), u_1(t), \mathbb{E}[u_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)])dt \\ &\quad + g(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)])dW(t) \\ &\quad + \tilde{g}(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)])dY(t), \\ x(0) &= a \in \mathbb{R}^n \end{cases} \quad (4.7)$$

with the cost functional

$$J(u_1(\cdot), \bar{u}_2(\cdot)) = \mathbb{E} \left[\int_0^T l(t, X(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)]) dt + m(X(T), \mathbb{E}[x(T)]) \right]. \quad (4.8)$$

Our optimal control problem is minimize $J(u_1(\cdot), \bar{u}_2(\cdot))$ over $u_1(\cdot) \in \mathcal{A}_1^S$, i.e., find $\bar{u}_1(\cdot) \in \mathcal{A}_1^S$ such that

$$J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \inf_{u_1(\cdot) \in \mathcal{A}_1} J(u_1(\cdot), \bar{u}_2(\cdot)). \quad (4.9)$$

Then for this case, it is easy to check that the Hamilton is $H(t, x, y, u_1, v_1, \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)], p, q, \tilde{q})$ and for the admissible control $\bar{u}_1(\cdot) \in \mathcal{A}_1^S$, the corresponding state process and the adjoint process is still $\bar{x}(\cdot)$ and $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot))$ respectively. Thus from the partial observed sufficient maximum principle for optimal control (see Theorem 3.1 in Tang and Meng (2016)), we conclude that $\bar{u}_1(\cdot)$ is the optimal control of the optimal control problem, i.e.,

$$J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) \leq J(u_1(\cdot), \bar{u}_2(\cdot)), \quad \text{for all } u_1(\cdot) \in \mathcal{A}_1^S,$$

and

$$J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \inf_{u_1(\cdot) \in \mathcal{A}_1^S} J(u_1(\cdot), \bar{u}_2(\cdot)).$$

The proof of (i) is complete.

(ii) This statement can be proved in a similar way as (i).

(iii) if both (i) and (ii) hold, then

$$J(\bar{u}_1(\cdot), u_2(\cdot)) \leq J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) \leq J(u_1(\cdot), \bar{u}_2(\cdot)),$$

for any $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1^S \times \mathcal{A}_2^S$. Thereby,

$$\begin{aligned} J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) &\leq \inf_{u_1(\cdot) \in \mathcal{A}_1^S} J(u_1(\cdot), \bar{u}_2(\cdot)) \\ &\leq \sup_{u_2(\cdot) \in \mathcal{A}_2^S} \left(\inf_{u_1(\cdot) \in \mathcal{A}_1^S} J(u_1(\cdot), u_2(\cdot)) \right). \end{aligned}$$

On the other hand,

$$\begin{aligned} J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) &\geq \sup_{u_2(\cdot) \in \mathcal{A}_2^S} J(\bar{u}_1(\cdot), u_2(\cdot)) \\ &\geq \inf_{u_1(\cdot) \in \mathcal{A}_1^S} \left(\sup_{u_2(\cdot) \in \mathcal{A}_2^S} J(u_1(\cdot), u_2(\cdot)) \right). \end{aligned}$$

Now, due to the inequality

$$\begin{aligned} &\inf_{u_1(\cdot) \in \mathcal{A}_1^S} \left(\sup_{u_2(\cdot) \in \mathcal{A}_2^S} J(u_1(\cdot), u_2(\cdot)) \right) \\ &\geq \sup_{u_2(\cdot) \in \mathcal{A}_2^S} \left(\inf_{u_1(\cdot) \in \mathcal{A}_1^S} J(u_1(\cdot), u_2(\cdot)) \right), \end{aligned}$$

we have

$$\begin{aligned} J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) &= \sup_{u_2 \in \mathcal{A}_2^S} \left(\inf_{u_1 \in \mathcal{A}_1^S} J(u_1, u_2) \right) \\ &= \inf_{u_1(\cdot) \in \mathcal{A}_1^S} \left(\sup_{u_2(\cdot) \in \mathcal{A}_2^S} J(u_1, u_2) \right). \end{aligned}$$

The proof is completed. \square

4.2 Necessary Conditions of Optimality

In this subsection we give the necessary Pontryagin maximum principle of Problem 2.3.

Theorem 4.2. *Let Assumption 2.1 and 2.2 be satisfied. Let $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ be an optimal open-loop control of Problem 2.3. Suppose that $\bar{x}(\cdot)$ is the state process of the system (2.10) corresponding to the admissible control $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$. Let $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot), \bar{r}(\cdot), \bar{R}(\cdot), \bar{\tilde{R}}(\cdot))$ be the unique solution of the adjoint equation (4.2) corresponding to $(\bar{u}_1(\cdot), \bar{u}_2(\cdot); \bar{x}(\cdot))$. Then the optimality conditions*

$$\left\langle \mathbb{E}[\bar{H}_{u_1}(t)|\mathcal{F}_t^Y] + \mathbb{E}[\bar{H}_{v_1}(t)], u_1 - \bar{u}_1(t) \right\rangle \geq 0 \quad (4.10)$$

and

$$\left\langle \mathbb{E}[\bar{H}_{u_2}(t)|\mathcal{F}_t^Y] + \mathbb{E}[\bar{H}_{v_2}(t)], u_2 - \bar{u}_2(t) \right\rangle \leq 0 \quad (4.11)$$

holds for any $(u_1, u_2) \in U_1 \times U_2$ and a.e. $(t, \omega) \in [0, T] \times \Omega$. Here using the notation (3.11), for $i = 1, 2$, we set

$$\bar{H}_{u_i}(t) = H_{u_i}(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}_1(t), \mathbb{E}[\bar{u}_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)]) \quad (4.12)$$

and

$$\bar{H}_{v_i}(t) = H_{v_i}(t, \bar{x}(t), \mathbb{E}[\bar{x}(t)], \bar{u}_1(t), \mathbb{E}[\bar{u}_1(t)], \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)]). \quad (4.13)$$

Proof: Since $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ is an optimal open-loop control, then $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ is an open-loop saddle point, i.e.,

$$J(\bar{u}_1(\cdot), u_2(\cdot)) \leq J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) \leq J(u_1(\cdot), \bar{u}_2(\cdot)). \quad (4.14)$$

So we have

$$J_1(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \min_{u_1(\cdot) \in \mathcal{A}_1^S} J_1(u_1(\cdot), \bar{u}_2(\cdot)) \quad (4.15)$$

and

$$J_2(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \max_{u_2(\cdot) \in \mathcal{A}_2^S} J_2(\bar{u}_1(\cdot), u_2(\cdot)). \quad (4.16)$$

By (4.15), $\bar{u}_1(\cdot)$ can be regarded as an optimal control of the optimal control problem where the controlled system is (4.7) and the cost functional is (4.8). Then for this case, it is easy to check that the Hamilton is $H(t, x, y, u_1, v_1, \bar{u}_2(t), \mathbb{E}[\bar{u}_2(t)], p, q, \tilde{q})$ and for the admissible control $\bar{u}_1(\cdot) \in \mathcal{A}_1^S$, the corresponding state process and the adjoint process is still $\bar{x}(\cdot)$ and $(\bar{p}(\cdot), \bar{q}(\cdot), \bar{\tilde{q}}(\cdot))$ respectively. Thus applying the partial necessary stochastic maximum principle for optimal control problems (see Theorem 3.5 in Tang and Meng (2016)), we can obtain (4.10). Similarly, from (4.16), we can obtain (4.11). The proof is completed.

5 An Example: A Linear Quadratic Differential Game Problem

In this section, we apply our stochastic maximum principle to solve a partial observed stochastic linear quadratic differential game problem. Let us make it more precise below. In this case, we assume the state system is the following linear mean-field SDE

$$\begin{cases} dX(t) = (A_1(t)X(t) + A_2(t)\mathbb{E}[X(t)] + B_{11}(t)u_1(t) + B_{12}(t)\mathbb{E}[u_1(t)] + B_{21}(t)u_2(t) + B_{22}(t)\mathbb{E}[u_2(t)])dt \\ \quad + (C_1(t)X(t) + C_2(t)\mathbb{E}[X(t)] + D_{11}(t)u_1(t) + D_{12}(t)\mathbb{E}[u_1(t)] + D_{21}(t)u_2(t) + D_{22}(t)\mathbb{E}[u_2(t)])dW(t) \\ \quad + (F_1(t)X(t) + F_2(t)\mathbb{E}[X(t)] + G_{11}(t)u_1(t) + G_{12}(t)\mathbb{E}[u_1(t)] + G_{21}(t)u_2(t) + G_{22}(t)\mathbb{E}[u_2(t)])dW^{(u_1, u_2)}(t), \\ x(0) = x \in \mathbb{R}^n, \end{cases} \quad (5.1)$$

with an observation

$$\begin{cases} dY(t) = h(t)dt + dW^{(u_1, u_2)}(t), \\ Y(0) = 0, \end{cases} \quad (5.2)$$

and the cost functional has the following quadratic form:

$$\begin{aligned}
J(u_1(\cdot), u_2(\cdot)) = & \mathbb{E}[\langle M, X(T) \rangle] + \mathbb{E} \left[\int_0^T \langle Q(s), X(s) \rangle ds \right] \\
& + \mathbb{E} \left[\int_0^T \langle N_{11}(s)u_1(s), u_1(s) \rangle ds \right] + \mathbb{E} \left[\int_0^T \langle N_{12}(s)\mathbb{E}[u_1(s)], \mathbb{E}[u_1(s)] \rangle ds \right] \\
& + \mathbb{E} \left[\int_0^T \langle N_{21}(s)u_2(s), u_2(s) \rangle ds \right] + \mathbb{E} \left[\int_0^T \langle N_{22}(s)\mathbb{E}[u_2(s)], \mathbb{E}[u_2(s)] \rangle ds \right].
\end{aligned} \tag{5.3}$$

In this case, our control process $(u_1(\cdot), u_2(\cdot))$ is said to be an admissible stochastic process if $(u_1(\cdot), u_2(\cdot)) \in M_{\mathcal{F}_Y}^2(0, T; \mathbb{R}^{k_1}) \times M_{\mathcal{F}_Y}^2(0, T; \mathbb{R}^{k_2})$. The set of all admissible controls is also denoted by $\mathcal{A}_1^S \times \mathcal{A}_2^S$. Note that there is no constraint on our control process, since it takes value in $\mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$. Now we make the basic assumptions on the coefficients.

Assumption 5.1. The matrix-valued functions $A_1, A_2, C_1, C_2, F_1, F_2, : [0, T] \rightarrow \mathbb{R}^{n \times n}; B_{11}, B_{12}, D_{11}, D_{12}, G_{11}, G_{12}, : [0, T] \rightarrow \mathbb{R}^{n \times k_1}; B_{21}, B_{22}, D_{21}, D_{22}, G_{21}, G_{22}, : [0, T] \rightarrow \mathbb{R}^{n \times k_2}; N_{11}, N_{12} : [0, T] \rightarrow \mathbb{R}^{k_1 \times k_1}; N_{21}, N_{22} : [0, T] \rightarrow \mathbb{R}^{k_2 \times k_2}; Q : [0, T] \rightarrow \mathbb{R}^n; h : [0, T] \rightarrow \mathbb{R}$ are uniformly bounded measurable functions. M is a vector in \mathbb{R}^n .

Assumption 5.2. The matrix-valued functions $N_{11}N_{11} + N_{12}$ are uniformly positive, i.e. for $\forall u_1 \in \mathbb{R}^{k_1}$ and a.s. $t \in [0, T]$, $\langle N_{11}(t)u_1, u_1 \rangle \geq \delta \langle u_1, u_1 \rangle$ and $\langle (N_{11}(t) + N_{12}(t))u_1, u_1 \rangle \geq \delta \langle u_1, u_1 \rangle$, for some positive constant δ . The matrix-valued functions $N_{21}N_{21} + N_{22}$ are uniformly negative, i.e. for $\forall u_2 \in \mathbb{R}^{k_2}$ and a.s. $t \in [0, T]$, $\langle N_{21}(t)u_2, u_2 \rangle \leq -\delta \langle u_2, u_2 \rangle$ and $\langle (N_{21}(t) + N_{22}(t))u_2, u_2 \rangle \leq -\delta \langle u_2, u_2 \rangle$, for some positive constant δ .

Our partial observed stochastic linear quadratic differential game control problem can be stated as follows.

Problem 5.1. Find an admissible open-loop control $(\bar{u}_1(\cdot), \bar{u}_2(\cdot))$ over $\mathcal{A}_1^S \times \mathcal{A}_2^S$ such that

$$J(\bar{u}_1(\cdot), \bar{u}_2(\cdot)) = \sup_{u_2(\cdot) \in \mathcal{A}_2^S} \left(\inf_{u_1(\cdot) \in \mathcal{A}_1^S} J(u_1(\cdot), u_2(\cdot)) \right) = \inf_{u_1(\cdot) \in \mathcal{A}_1^S} \left(\sup_{u_2(\cdot) \in \mathcal{A}_2^S} J(u_1(\cdot), u_2(\cdot)) \right), \tag{5.4}$$

subject to (5.1), (5.2) and (5.3).

It is easy to check that under Assumptions 5.1 and 5.2, if we set

$$\begin{aligned}
b(t, x, y, u_1, v_1, u_2, v_2) &= A_1(t)x + A_2(t)y + B_{11}(t)u_1 + B_{12}(t)v_1 + B_{21}(t)u_2 + B_{22}(t)v_2, \\
g(t, x, y, u_1, v_1, u_2, v_2) &= C_1(t)x + C_2(t)y + D_{11}(t)u_1 + D_{12}(t)v_1 + D_{21}(t)u_2 + D_{22}(t)v_2, \\
\tilde{g}(t, x, y, u, v) &= F_1(t)x + F_2(t)y + G_{11}(t)u_1 + G_{12}(t)v_1 + G_{21}(t)u_2 + G_{22}(t)v_2, \\
h(t, x, y, u_1, v_1, u_2, v_2) &= h(t), m(x, y) = \langle M, x \rangle, \\
l(t, x, y, u_1, v_1, u_2, v_2) &= \langle Q_1(t), x \rangle + \langle N_{11}(t)u_1, u_1 \rangle + \langle N_{12}(t)v_1, v_1 \rangle + \langle N_{21}(t)u_2, u_2 \rangle + \langle N_{22}(t)v_2, v_2 \rangle.
\end{aligned} \tag{5.5}$$

Problem 5.1 can be regarded as a special case of Problem 2.3 and Assumptions 2.1 and 2.2 for (5.5) hold. Thus Theorem 4.1 and 4.2 can be applied to solve Problem 5.1. In this case, the Hamiltonian becomes

$$\begin{aligned}
H(t, x, y, u, v, p, q, \tilde{q}) &= \langle p, A_1(t)x + A_2(t)y + B_{11}(t)u_1 + B_{12}(t)v_1 + B_{21}(t)u_2 + B_{22}(t)v_2 \\
&\quad - h(t)(F_1(t)x + F_2(t)y + G_{11}(t)u_1 + G_{12}(t)v_1 + G_{21}(t)u_2 + G_{22}(t)v_2) \rangle \\
&\quad + \langle q, C_1(t)x + C_2(t)y + D_{11}(t)u_1 + D_{12}(t)v_1 + D_{21}(t)u_2 + D_{22}(t)v_2 \rangle \\
&\quad + \langle \tilde{q}, F_1(t)x + F_2(t)y + G_{11}(t)u_1 + G_{12}(t)v_1 + G_{21}(t)u_2 + G_{22}(t)v_2 \rangle \\
&\quad + \langle Q_1, x \rangle + \langle N_{11}u_1, u_1 \rangle + \langle N_{12}v_1, v_1 \rangle + \langle N_{21}u_2, u_2 \rangle + \langle N_{22}v_2, v_2 \rangle.
\end{aligned} \tag{5.6}$$

For any admissible pair $(u_1(\cdot), u_2(\cdot); x(\cdot))$, the corresponding adjoint equation becomes

$$\begin{cases} dp(t) &= - \left[(A_1^\top(t) - h(t)F_1^\top(t))p(t) + (A_2^\top(t) - h(t)F_2^\top(t))\mathbb{E}[p(t)] + C_1^\top(t)q(t) \right. \\ &\quad \left. + C_2^\top(t)\mathbb{E}[q(t)] + F_1^\top(t)\tilde{q}(t) + F_2^\top(t)\mathbb{E}[\tilde{q}(t)] + Q(t) \right] dt \\ p(T) &= M. \end{cases} \quad (5.7)$$

In the following, we will prove Problem 5.1 admits at least an open-loop saddle point $(\bar{u}(\cdot), \bar{u}_2(\cdot)) \in \mathcal{A}_1^S \times \mathcal{A}_2^S$. To the end, we need the following basic result on a saddle point in convex analysis theory.

Lemma 5.2. (*Proposition 2.4 of Chapter VI in Ekeland and Témam (1976)*). *Let \mathcal{A} and \mathcal{B} be two reflex Banach space. Assume that a function L of $\mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}$ satisfies*

$$\forall u \in \mathcal{A}, p \rightarrow L(u, p)$$

is concave upper-semi continuous and

$$\forall p \in \mathcal{B}, u \rightarrow L(u, p)$$

is convex lower-semi continuous. Moreover, there exist $u_0 \in \mathcal{A}$ and $p_0 \in \mathcal{B}$ such that

$$\lim_{\|u\| \rightarrow \infty} L(u, p_0) = +\infty$$

and

$$\lim_{\|p\| \rightarrow \infty} L(u_0, p) = -\infty.$$

Then L possesses at least one saddle point on L .

Theorem 5.3. *Let Assumptions 5.1 and 5.2 be satisfied. Then Problem 5.1 has at least an saddle point $(\bar{u}(\cdot), \bar{u}_2(\cdot)) \in \mathcal{A}_1^S \times \mathcal{A}_2^S$ and*

$$\sup_{u_2(\cdot) \in \mathcal{A}_2} \left(\inf_{u_1(\cdot) \in \mathcal{A}_1} J(u_1(\cdot), u_2(\cdot)) \right) = \inf_{u_1(\cdot) \in \mathcal{A}_1} \left(\sup_{u_2(\cdot) \in \mathcal{A}_2} J(u_1(\cdot), u_2(\cdot)) \right). \quad (5.8)$$

Proof. Since $\mathcal{A}_1^S = M_{\mathcal{H}_Y}^2(0, T; \mathbb{R}^{k_1})$ and $\mathcal{A}_2^S = M_{\mathcal{H}_Y}^2(0, T; \mathbb{R}^{k_2})$ are Hilbert spaces thus reflexive Banach spaces. Indeed, by the a priori estimate for the state process in Lemma 1.4 in Tang and Meng (2016), over $\mathcal{A}_1^S \times \mathcal{A}_2^S$, we can show that the cost functional $J(u_1(\cdot), u_2(\cdot))$ is continuous with respect to $(u_1(\cdot), u_2(\cdot))$ and hence lower-semi continuous with respect to $u_1(\cdot)$ and upper-semi continuous with respect to $u_2(\cdot)$, respectively. Since the weighting matrices in the cost functional are not random, from the definition of $J(u_1(\cdot), u_2(\cdot))$ (see (5.3)) and by a simple calculation, we can get that

$$\begin{aligned} J(u_1(\cdot), u_2(\cdot)) &= \mathbb{E} \left[\int_0^T (\langle N_1(t)(u(t) - \mathbb{E}[u(t)]), u(t) - \mathbb{E}[u(t)] \rangle + \langle (N_1(t) + N_2(t))\mathbb{E}[u(t)], \mathbb{E}[u(t)] \rangle) dt \right] \\ &\quad + \mathbb{E} \left[\langle M, X(T) \rangle \right] + \mathbb{E} \left[\int_0^T \left(\langle Q(t), X(t) \rangle \right) dt \right] \\ &\quad + \mathbb{E} \left[\int_0^T \langle N_{21}(s)u_2(s), u_2(s) \rangle ds \right] + \mathbb{E} \left[\int_0^T \langle N_{22}(s)\mathbb{E}[u_2(s)], \mathbb{E}[u_2(s)] \rangle ds \right]. \end{aligned} \quad (5.9)$$

Thus the cost functional $J(u_1(\cdot), u_2(\cdot))$ is convex with respect to $u_1(\cdot)$ over \mathcal{A}_1^S from the nonnegativity of the $N_{11}, N_{11} + N_{12}$. Furthermore, it follows from the uniformly strictly positivity of $N_{11}, N_{11} + N_{12}$ and the a priori

estimate for the state process, that

$$\begin{aligned}
J(u_1(\cdot), u_2(\cdot)) &\geq \delta \mathbb{E} \left[\int_0^T \langle u_1(t) - \mathbb{E}[u_1(t)], u_1(t) - \mathbb{E}[u_1(t)] \rangle dt \right] + \delta \mathbb{E} \left[\int_0^T \langle \mathbb{E}[u_1(t)], \mathbb{E}[u_1(t)] \rangle dt \right] - K \|u_1(\cdot)\|_{\mathcal{A}_1^S} \\
&\quad + \mathbb{E} \left[\int_0^T \langle N_{21}(s) u_2(s), u_2(s) \rangle ds \right] + \mathbb{E} \left[\int_0^T \langle N_{22}(s) \mathbb{E}[u_2(s)], \mathbb{E}[u_2(s)] \rangle ds \right]. \\
&= \delta \|u_1(\cdot)\|_{\mathcal{A}_1^S}^2 - K \|u_1(\cdot)\|_{\mathcal{A}_1^S} \\
&\quad + \mathbb{E} \left[\int_0^T \langle N_{21}(s) u_2(s), u_2(s) \rangle ds \right] + \mathbb{E} \left[\int_0^T \langle N_{22}(s) \mathbb{E}[u_2(s)], \mathbb{E}[u_2(s)] \rangle ds \right],
\end{aligned} \tag{5.10}$$

which implies

$$\lim_{\|u_1(\cdot)\|_{\mathcal{A}_1^S} \rightarrow +\infty} J(u_1(\cdot), u_2(\cdot)) = +\infty.$$

On the other hand, we have

$$\begin{aligned}
J(u_1(\cdot), u_2(\cdot)) &= \mathbb{E} \left[\int_0^T (\langle N_{21}(t)(u_2(t) - \mathbb{E}[u_2(t)]), u_2(t) - \mathbb{E}[u_2(t)] \rangle + \langle (N_{21}(t) + N_{22}(t)) \mathbb{E}[u_2(t)], \mathbb{E}[u_2(t)] \rangle) dt \right] \\
&\quad + \mathbb{E} \left[\langle M, X(T) \rangle \right] + \mathbb{E} \left[\int_0^T \langle Q(t), X(t) \rangle dt \right] \\
&\quad + \mathbb{E} \left[\int_0^T \langle N_{11}(s) u_1(s), u_1(s) \rangle ds \right] + \mathbb{E} \left[\int_0^T \langle N_{12}(s) \mathbb{E}[u_1(s)], \mathbb{E}[u_1(s)] \rangle ds \right].
\end{aligned} \tag{5.11}$$

Thus the cost functional $J(u_1(\cdot), u_2(\cdot))$ over \mathcal{A}_2^S is concave with respect to $u_2(\cdot)$ from the negativity of the $N_{21}, N_{21} + N_{22},$. Furthermore, it follows from the uniformly strictly negativity of $N_{21}, N_{21} + N_{22},$, that

$$\begin{aligned}
J(u_1(\cdot), u_2(\cdot)) &\leq -\delta \mathbb{E} \left[\int_0^T \langle u_2(t) - \mathbb{E}[u_2(t)], u_2(t) - \mathbb{E}[u_2(t)] \rangle dt \right] + -\delta \mathbb{E} \left[\int_0^T \langle \mathbb{E}[u_2(t)], \mathbb{E}[u_2(t)] \rangle dt \right] + K \|u_2(\cdot)\|_{\mathcal{A}_1^S} \\
&\quad + \mathbb{E} \left[\int_0^T \langle N_{11}(s) u_1(s), u_1(s) \rangle ds \right] + \mathbb{E} \left[\int_0^T \langle N_{12}(s) \mathbb{E}[u_1(s)], \mathbb{E}[u_1(s)] \rangle ds \right] \\
&= -\delta \|u_2(\cdot)\|_{\mathcal{A}_2^S}^2 + K \|u_2(\cdot)\|_{\mathcal{A}_2^S} \\
&\quad + \mathbb{E} \left[\int_0^T \langle N_{11}(s) u_1(s), u_1(s) \rangle ds \right] + \mathbb{E} \left[\int_0^T \langle N_{12}(s) \mathbb{E}[u_1(s)], \mathbb{E}[u_1(s)] \rangle ds \right],
\end{aligned} \tag{5.12}$$

which implies

$$\lim_{\|u_2(\cdot)\|_{\mathcal{A}_2^S} \rightarrow +\infty} J(u_1(\cdot), u_2(\cdot)) = -\infty.$$

In summary, by Lemma 5.2, Problem 5.1 has at least an saddle point $(\bar{u}(\cdot), \bar{u}_2(\cdot)) \in \mathcal{A}_1^S \times \mathcal{A}_2^S$. The proof is complete. \square

In the following, applying the maximum principle to our LQ differential game problem, we give the dual presentation of the optimal control in terms of the corresponding adjoint process.

Theorem 5.4. *Let Assumptions 5.1 and 5.2 be satisfied. Then, a necessary and sufficient condition for an admissible pair $(u_1(\cdot), u_2(\cdot); x(\cdot))$ to be an optimal pair of Problem (LQ) is the control $(u_1(\cdot), u_2(\cdot))$ satisfies*

$$\begin{aligned}
&2N_{11}(t)u(t) + 2N_{12}(t)\mathbb{E}[u_1(t)] + (B_1^\top(t) - h(t)G_{11}^\top(t))\mathbb{E}[p(t)|\mathcal{F}_t^Y] + (B_{12}^\top(t) - h(t)G_{12}^\top(t))\mathbb{E}[p(t)] \\
&\quad + D_{12}^\top(t)\mathbb{E}[q(t)|\mathcal{F}_t^Y] + D_{12}^\top(t)\mathbb{E}[q(t)] = 0, \quad a.e.a.s.,
\end{aligned} \tag{5.13}$$

and

$$\begin{aligned} & 2N_{21}(t)u(t) + 2N_{22}(t)\mathbb{E}[u(t)] + (B_1^\top(t) - h(t)G_{21}^\top(t))\mathbb{E}[p(t)|\mathcal{F}_t^Y] + (B_{22}^\top(t) - h(t)G_{22}^\top(t))\mathbb{E}[p(t)] \\ & + D_{21}^\top(t)\mathbb{E}[q(t)|\mathcal{F}_t^Y] + D_{22}^\top(t)\mathbb{E}[q(t)] = 0, \quad a.e.a.s., \end{aligned} \quad (5.14)$$

where $(p(\cdot), q(\cdot), \tilde{q}(\cdot))$ is the solution to the adjoint equation (5.7) corresponding to $(u(\cdot), X(\cdot))$.

Proof. For the necessary part, let $(u_1(\cdot), u_2(\cdot), x(\cdot))$ be an optimal pair associated with the adjoint process $(p(\cdot), q(\cdot), \tilde{q}(\cdot))$. Since there is no constraints on the control processes, then from the necessary optimality conditions (4.10) and (4.11) (see Theorem 4.2), we get that for $i = 1, 2$,

$$\begin{aligned} & \mathbb{E} \left[H_{u_i}(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)], p(t), q(t), \tilde{q}(t)) | \mathcal{F}_t^Y \right] \\ & + \mathbb{E} \left[H_{v_i}(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)], p(t), q(t), \tilde{q}(t)) \right] = 0, \end{aligned} \quad (5.15)$$

which leads to (5.14) and (5.13) (recalling the definition (5.6) of Hamiltonian H).

For the sufficient part, let $(u(\cdot), X(\cdot))$ be an admissible pair associated with the adjoint process $(p(\cdot), q(\cdot), \tilde{q}(\cdot))$ and assume the conditions (5.1) and (5.14) hold. From the definition of H (see (5.6)), the conditions (5.13) and (5.14) implies (5.15) holds. Thus, since any admissible control is \mathcal{F}_t^Y -adapted process. by (5.15), for any other admissible control $v_i(\cdot) \in \mathcal{A}_i^S, i = 1, 2$, from the property of conditional expectation, we have

$$\begin{aligned} & \mathbb{E} \left[\left\langle v_i(t) - u_i(t), H_{u_i}(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)], p(t), q(t), \tilde{q}(t)) \right. \right. \\ & \quad \left. \left. + \mathbb{E} \left[H_{v_i}(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)], p(t), q(t), \tilde{q}(t)) \right] \right\rangle \right] \\ & = \mathbb{E} \left[\left\langle v_i(t) - \bar{u}_i(t), \mathbb{E} \left[H_{u_i}(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)], p(t), q(t), \tilde{q}(t)) | \mathcal{F}_t^Y \right] \right. \right. \\ & \quad \left. \left. + \mathbb{E} \left[H_{v_i}(t, x(t), \mathbb{E}[x(t)], u_1(t), \mathbb{E}[u_1(t)], u_2(t), \mathbb{E}[u_2(t)], p(t), q(t), \tilde{q}(t)) \right] \right\rangle \right] \\ & = 0, \end{aligned} \quad (5.16)$$

which implies that the conditions (4.8) and (4.5) in Theorem 4.1 holds. Moreover, under Assumptions 5.1 and 5.2, it is easy to check that all other conditions of (i) and (ii) in Theorem 4.1 are satisfied. Therefore, by (iii) in Theorem 4.1, we conclude that $(u_1(\cdot), u_2(\cdot); x(\cdot))$ is an optimal loop-open control pair. The proof is complete. \square

From the above result, we see that if Problem 2.3 admits an open-loop saddle point, then the following FBSDE admits an adapted solution $(u(\cdot), x(\cdot), p(\cdot), q(\cdot), \tilde{q}(\cdot))$

$$\begin{cases} dX(t) = & (A_1(t)X(t) + A_2(t)\mathbb{E}[X(t)] + B_{11}(t)u_1(t) + B_{12}(t)\mathbb{E}[u_1(t)] + B_{21}(t)u_2(t) + B_{22}(t)\mathbb{E}[u_2(t)])dt \\ & + (C_1(t)X(t) + C_2(t)\mathbb{E}[X(t)] + D_{11}(t)u_1(t) + D_{12}(t)\mathbb{E}[u_1(t)] + D_{21}(t)u_2(t) + D_{22}(t)\mathbb{E}[u_2(t)])dW(t) \\ & + (F_1(t)X(t) + F_2(t)\mathbb{E}[X(t)] + G_{11}(t)u_1(t) + G_{12}(t)\mathbb{E}[u_1(t)] + G_{21}(t)u_2(t) + G_{22}(t)\mathbb{E}[u_2(t)])dW^{(u_1, u_2)}(t), \\ dY(t) = & h(t)dt + dW^{(u_1, u_2)}(t), \\ dp(t) = & - \left[(A_1^\top(t) - h(t)F_1^\top(t))p(t) + (A_2^\top(t) - h(t)F_2^\top(t))\mathbb{E}[p(t)] + C_1^\top(t)q(t) \right. \\ & \left. + C_2^\top(t)\mathbb{E}[q(t)] + F_1^\top(t)\tilde{q}(t) + F_2^\top(t)\mathbb{E}[\tilde{q}(t)] + Q(t) \right] dt + q(t)dW(t) + \tilde{q}(t)dY(t), \\ x(0) = & x, p(T) = M, Y(0) = 0 \\ 2N_{i1}(t)u(t) & + 2N_{i2}(t)\mathbb{E}[u_i(t)] + (B_{i1}^\top(t) - h(t)G_{i1}^\top(t))\mathbb{E}[p(t)|\mathcal{F}_t^Y] + (B_{i2}^\top(t) - h(t)G_{i2}^\top(t))\mathbb{E}[p(t)] \\ & + D_{i2}^\top(t)\mathbb{E}[q(t)|\mathcal{F}_t^Y] + D_{i2}^\top(t)\mathbb{E}[q(t)] = 0, i = 1, 2, \quad a.e.a.s. \end{cases} \quad (5.17)$$

Then by Theorem 5.4, we can directly obtain the following equivalence between the solvability of optimality system (5.17) and the existence of the optimal open-loop control of Problem 5.1.

Corollary 5.5. *Let Assumptions 5.1 and 5.2 be satisfied. Then, a necessary and sufficient condition for that the optimality system (5.17) has a solution $(u_1(\cdot), u_2(\cdot), x(\cdot), p(\cdot), q(\cdot), \tilde{q}(\cdot)) \in M_{\mathcal{F}^Y}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}^Y}^2(0, T; \mathbb{R}^n) \times S_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times S_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}^n) \times M_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$ is that $(u_1(\cdot), u_2(\cdot); x(\cdot))$ is an optimal open-loop pair of Problem 5.1.*

Remark 5.1. In summary, the optimality system (5.17) completely characterizes the optimal open-loop control of Problem 5.1. Therefore, solving Problem 5.1 is equivalent to solving the optimality system, moreover, the optimal open-loop control can be given by (5.13) and (5.14) which implies that the optimal loop-open control $(u_1(\cdot), u_2(\cdot))$ has the following explicit dual presentation

$$\begin{aligned} \bar{u}_1(t) = & -\frac{1}{2}N_{11}^{-1}(t)\left\{(B_{11}^\top(t) - h(t)G_{11}^\top(t))\mathbb{E}[p(t)|\mathcal{F}_t^Y] + (B_{12}^\top(t) - h(t)G_{12}^\top(t))\mathbb{E}[p(t)] \right. \\ & + D_{11}^\top(t)\mathbb{E}[q(t)|\mathcal{F}_t^Y] + D_{12}^\top(t)\mathbb{E}[q(t)] \\ & + N_{12}(t)(N_{11}(t) + N_{12}(t))^{-1}\left[(B_{11}(t) + B_{12}(t) - h(t)G_{11}(t) - h(t)G_{12}(t))^\top\mathbb{E}[p(t)] \right. \\ & \left. \left. + (D_{11}(t) + D_{12}(t))^\top\mathbb{E}[q(t)]\right]\right\}, \quad a.e.a.s. \end{aligned} \quad (5.18)$$

and

$$\begin{aligned} \bar{u}_2(t) = & -\frac{1}{2}N_{21}^{-1}(t)\left\{(B_{21}^\top(t) - h(t)G_{21}^\top(t))\mathbb{E}[p(t)|\mathcal{F}_t^Y] + (B_{22}^\top(t) - h(t)G_{22}^\top(t))\mathbb{E}[p(t)] \right. \\ & + D_{21}^\top(t)\mathbb{E}[q(t)|\mathcal{F}_t^Y] + D_{22}^\top(t)\mathbb{E}[q(t)] \\ & + N_{22}(t)(N_{21}(t) + N_{22}(t))^{-1}\left[(B_{21}(t) + B_{22}(t) - h(t)G_{21}(t) - h(t)G_{22}(t))^\top\mathbb{E}[p(t)] \right. \\ & \left. \left. + (D_{21}(t) + D_{22}(t))^\top\mathbb{E}[q(t)]\right]\right\}, \quad a.e.a.s. \end{aligned} \quad (5.19)$$

6 Conclusion

In this paper, we have proved partial observed stochastic maximum principle for the stochastic differential games driven by mean-field stochastic differential equations. As an application, some partial observed linear quadratic stochastic differential game problem of mean-field type is discussed and the existence of the open-loop saddle is obtained and the optimal control process is characterized explicitly by the adjoint process. Our main results could be seen as an extension of the stochastic optimal control problem studied in Tang and Meng (2016), to the two-person zero-sum stochastic differential game problem.

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